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# A NEW TYPE OF SOLUTION OF LAPLACE'S EQUATION.

BY H. BATEMAN.

1. In the work of classifying solutions\* of the potential equation  $\nabla^2 V = 0$ , the problem of constructing solutions involving arbitrary functions deserves considerable attention.

There are really two problems of this kind, but they are intimately related with one another. In the first problem the aim is to find potential functions of the form †

$$V = f(X, Y), \quad (1)$$

in which  $X$  and  $Y$  are certain functions of  $x, y, z$  and  $f$  satisfies a partial differential equation in the two variables  $X, Y$ .

This problem has been solved by Levi Civita and generalized by Amaldi‡ who considers solutions of the form

$$V = uf(X, Y), \quad (2)$$

where  $f$  satisfies a partial differential equation. This generalized problem can be discussed by means of a relation of the form§

$$(dx^2 + dy^2 + dz^2)(p^2 + q^2 + r^2) - (pdx + qdy + rdz)^2 = AdX^2 + 2HdXdY + BdY^2$$

in which  $A, H, B$  are functions of  $X$  and  $Y$ .

In the second problem the aim is to find solutions of the form

$$V = u f(\theta), \quad (3)$$

where  $f$  is an arbitrary function with a finite second derivative and  $u, \theta$  are certain functions of  $x, y, z$ . It is known that when a solution of this form exists, the parameter  $\theta$  must satisfy the differential equation of the characteristics

$$\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2 = 0 \quad (4)$$

and this fact has been used by Forsyth|| to obtain a class of solutions of the form (3). His result is as follows:

\* This work has been commenced by Levi Civita, *Torino Memoirs*, vol. XLIX (1899), pp. 105–152. A different classification is given in Kelvin-Tait's *Natural Philosophy*.

† Volterra has called these functions binary potentials. "Sopra alcuni problemi della teoria del potenziale," *Ann. Sc. Normale di Pisa*, 1883.

‡ *Rend. Palermo*, vol. 16 (1902), pp. 1–45.

§ *Cambr. Phil. Trans.*, vol. 21 (1910), pp. 257–280.

|| *Mess. Math.*, vol. XXVII (1898), pp. 99–118; *Phil. Trans. A.*, vol. CXCI (1898). The first part of the theorem is due to Jacobi, *Werke*, Bd. 2, p. 208.

Let  $p, q, r$  be three functions of  $\theta$  such that

$$p^2 + q^2 + r^2 = 0$$

and let  $\theta$  be defined in terms of  $x, y, z$  by means of the equation

$$\theta = px + qy + rz;$$

then  $V = f(\theta)$  is a solution of  $\nabla^2 V = 0$  and a second solution is given by

$$V = \frac{g(\theta)}{1 - xp'(\theta) - yq'(\theta) - zr'(\theta)}.$$

Some developments of this theorem have been given by Burnside\* and Bromwich.† The latter obtains a class of solutions of the form

$$V = Xf(u) + Yf'(u).$$

Forsyth has obtained some analogous theorems for the equation of wave motion, showing in particular that if  $\alpha, \beta, \gamma, \sigma$  are four functions of  $\theta$  satisfying the equation

$$\sigma^2 = \alpha^2 + \beta^2 + \gamma^2$$

and  $\theta$  be defined in terms of  $x, y, z, t$  by the equation

$$F(\theta) = x\alpha(\theta) + y\beta(\theta) + z\gamma(\theta) - t\sigma(\theta),$$

then  $V = f(\theta)$  is a solution of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial t^2} \quad (5)$$

and

$$\frac{X(\theta)}{F'(\theta) - x\alpha'(\theta) - y\beta'(\theta) - z\gamma'(\theta) + t\sigma'(\theta)}$$

is another solution of this equation.

A solution analogous to this is well known in the theory of electrons, as it occurs in the specification of an electromagnetic field with a simple singularity moving through the æther.‡ It is obtained as follows: Let  $\theta$  be defined by the equation

$$[x - \xi(\theta)]^2 + [y - \eta(\theta)]^2 + [z - \zeta(\theta)]^2 = [t - \tau(\theta)]^2; \quad (6)$$

then if

$$M = \xi'(\theta)[x - \xi(\theta)] + \eta'(\theta)[y - \eta(\theta)] + \zeta'(\theta)[z - \zeta(\theta)] - \tau'(\theta)[t - \tau(\theta)]$$

the function  $V = f(\theta)/M$  satisfies (5) and  $\theta$  is a solution of the equation

$$\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2 = \left(\frac{\partial \theta}{\partial t}\right)^2. \quad (7)$$

\* *Mess. Math.*, vol. XXVII (1898), pp. 138-146. See also Schottky, *Berl. Ber.* (1909.)

† *Proc. London Math. Soc.*, Ser. 1, vol. 30 (1899).

‡ A. Liénard, *L'éclairage électrique* (1898), pp. 5, 53, 106; E. Wiechert, *Archives néerlandaises* (2), 5 (1900), p. 54; A. W. Conway, *Proc. London Math. Soc.*, Ser. 2, vol. 1, p. 154, 1903; H. Bateman, *Ibid.*, 1911.

The corresponding solution of Laplace's equation does not seem to have been given hitherto, probably because it is of a slightly different form. If we define  $\theta$  in terms of  $x, y, z$  by the equation\*

$$[x - \xi(\theta)]^2 + [y - \eta(\theta)]^2 + [z - \zeta(\theta)]^2 = 0 \quad (8)$$

and write

$$M = \xi'(\theta)[x - \xi(\theta)] + \eta'(\theta)[y - \eta(\theta)] + \zeta'(\theta)[z - \zeta(\theta)]$$

it appears that  $1/M^k$  is not generally a solution of Laplace's equation for any value of  $k$ . To obtain a solution in this case we proceed as follows: Let  $l, m, n$  be three functions of  $\theta$  satisfying the equations

$$\begin{aligned} l\xi'(\theta) + m\eta'(\theta) + n\zeta'(\theta) &= 0, \\ l^2 + m^2 + n^2 &= 0, \end{aligned} \quad (9)$$

and let  $w$  be defined by the equation

$$w = l(x - \xi) + m(y - \eta) + n(z - \zeta). \quad (10)$$

Differentiating we find that

$$\frac{\partial w}{\partial x} = l + K \frac{\partial \theta}{\partial x}, \quad \frac{\partial w}{\partial y} = m + K \frac{\partial \theta}{\partial y}, \quad \frac{\partial w}{\partial z} = n + K \frac{\partial \theta}{\partial z}$$

where

$$K = l'(x - \xi) + m'(y - \eta) + n'(z - \zeta).$$

These equations combined with (4) and (9) give

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 = 2K \left[ l \frac{\partial \theta}{\partial x} + m \frac{\partial \theta}{\partial y} + n \frac{\partial \theta}{\partial z} \right]. \quad (11)$$

The second derivatives of  $w$  are given by formulæ of the type

$$\frac{\partial^2 w}{\partial x^2} = 2l' \frac{\partial \theta}{\partial x} + N \left( \frac{\partial \theta}{\partial x} \right)^2 + K \frac{\partial^2 \theta}{\partial x^2},$$

where the coefficient  $N$  is the same in each. Adding these we get

$$\nabla^2 w = 2 \left[ l' \frac{\partial \theta}{\partial x} + m' \frac{\partial \theta}{\partial y} + n' \frac{\partial \theta}{\partial z} \right] + K \nabla^2 \theta. \quad (12)$$

Now

$$x - \xi = [\xi'(x - \xi) + \eta'(y - \eta) + \zeta'(z - \zeta)] \frac{\partial \theta}{\partial x} = M \frac{\partial \theta}{\partial x}.$$

Differentiating we find that

$$1 - \xi' \frac{\partial \theta}{\partial x} = M \frac{\partial^2 \theta}{\partial x^2} + \left( \xi' + L \frac{\partial \theta}{\partial x} \right) \frac{\partial \theta}{\partial x}$$

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\* This makes  $\theta$  a solution of equation (4).

and two similar equations with the same coefficient  $L$ . Adding these we get

$$3 - 2 \left( \xi' \frac{\partial \theta}{\partial x} + \eta' \frac{\partial \theta}{\partial y} + \zeta' \frac{\partial \theta}{\partial z} \right) = M \nabla^2 \theta.$$

But the equations

$$x - \xi = M \frac{\partial \theta}{\partial x}$$

give

$$\xi' \frac{\partial \theta}{\partial x} + \eta' \frac{\partial \theta}{\partial y} + \zeta' \frac{\partial \theta}{\partial z} = 1;$$

therefore

$$M \nabla^2 \theta = 1.$$

Equations (11) and (12) may now be written in the form

$$\left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 = \frac{2Kw}{M}, \quad \nabla^2 w = \frac{3K}{M},$$

and consequently

$$2w \nabla^2 w - 3 \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] = 0.$$

This means that the function

$$u = \frac{1}{\sqrt{w}}$$

is a solution of  $\nabla^2 u = 0$ .

It should be noticed that equations (9) give two sets of values for the *ratios* of the quantities  $l, m, n$ . If one appropriate set of values of  $l, m, n$  is known, another set can be obtained by multiplying each quantity by  $f(\theta)$ ; the corresponding function  $w$  becomes multiplied by  $f(\theta)$  and we infer that the function

$$V = \frac{g(\theta)}{\sqrt{w}}, \quad g(\theta) = \frac{1}{\sqrt{f(\theta)}}$$

is a solution of  $\nabla^2 V = 0$ .

If  $X, Y, Z$  are interpreted as current coördinates of a point, the plane whose equation is

$$l[X - \xi] + m[Y - \eta] + n[Z - \zeta] = 0$$

is a common tangent plane of the curve

$$X = \xi(\lambda), \quad Y = \eta(\lambda), \quad Z = \zeta(\lambda),$$

and the circle at infinity. It is easy to see that the function  $u = 1/\sqrt{w}$  becomes infinite at points of this curve; hence we have constructed a potential function whose singularities lie along an arbitrary curve. To illustrate the theorem let us first of all define  $\theta$  by the equation

$$x^2 + y^2 + (z - \theta)^2 = 0;$$

this gives

$$\theta = z \pm i\rho. \quad (\rho^2 = x^2 + y^2)$$

The equations for determining  $l, m, n$ , are

$$n = 0, \quad l^2 + m^2 = 0$$

hence we may take  $w = x \pm iy$  and we find that

$$\frac{1}{\sqrt{x \pm iy}} f(z \pm i\rho)$$

is a solution of Laplace's equation. Putting,

$$x = \rho \cos \phi, \quad y = \rho \sin \phi,$$

we obtain real solutions of the form \*

$$\rho^{-\frac{1}{2}} \cos \frac{\phi}{2} [f(z + i\rho) + f(z - i\rho)],$$

$$\rho^{-\frac{1}{2}} \sin \frac{\phi}{2} [f(z + i\rho) + f(z - i\rho)],$$

which are periodic in  $\phi$  with the period  $4\pi$ . Next, let us define  $\theta$  by the equation

$$(x - a \cos \theta)^2 + (y - a \sin \theta)^2 + z^2 = 0.$$

The equations for determining  $l, m, n$  are

$$l \sin \theta - m \cos \theta = 0, \quad l^2 + m^2 + n^2 = 0.$$

Taking  $l = \cos \theta, m = \sin \theta, n = i$ , we have

$$w = \cos \theta (x - a \cos \theta) + \sin \theta (y - a \sin \theta) + iz;$$

hence

$$2aw = x^2 + y^2 + (z + ia)^2,$$

and we conclude that the function

$$V = \frac{f(\theta)}{\sqrt{x^2 + y^2 + (z + ia)^2}}$$

is a solution of Laplace's equation.

Putting  $x = \rho \cos \phi, y = \rho \sin \phi$  and introducing the coördinates  $\sigma, \psi$  defined by the equations

$$\rho = \frac{a \sinh \sigma}{\cosh \sigma - \cos \psi}, \quad z = \frac{a \sin \psi}{\cosh \sigma - \cos \psi}$$

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\* These are well known.

we find that

$$\frac{1}{[\rho^2 + (z + ia)^2]^{\frac{1}{2}}} = e^{-\frac{i\psi}{2}} \sqrt{\cosh \sigma - \cos \psi},$$

$$\cos(\theta - \phi) = \frac{a^2 + z^2 + \rho^2}{2a\rho} = \coth \sigma, \quad e^{i(\theta - \phi)} = \tanh \frac{\sigma}{2}.$$

Hence the solution

$$\frac{e^{im\theta}}{[\rho^2 + (z + ia)^2]^{\frac{1}{2}}}$$

is transformed into

$$e^{-\frac{i\psi}{2}} \sqrt{\cosh \sigma - \cos \psi} \cdot e^{im\phi} \tanh^m \frac{\sigma}{2},$$

and we have real solutions of the form

$$\sqrt{\cosh \sigma - \cos \psi} \cos \frac{\psi}{2} \cdot \cos(m\phi + \epsilon) \left( \tanh \frac{\sigma}{2} \right)^m,$$

$$\sqrt{\cosh \sigma - \cos \psi} \sin \frac{\psi}{2} \cdot \cos(m\phi + \epsilon) \left( \tanh \frac{\sigma}{2} \right)^m,$$

It is well known that Laplace's equation is satisfied by a function of the form

$$V = \sqrt{\cosh \sigma - \cos \psi} \cos n\psi \cos m\phi P_{n-\frac{1}{2}}^m(\cosh \sigma);$$

the solutions just obtained belong to this class, for the differential equation for the associated Legendre function  $P_{n-\frac{1}{2}}^m(\cosh \sigma)$  is satisfied by  $(\tanh \sigma/2)^m$  when  $n = \frac{1}{2}$ .

It should be noticed that a solution of the equation

$$\left( \frac{\partial \theta}{\partial x} \right)^2 + \left( \frac{\partial \theta}{\partial y} \right)^2 + \left( \frac{\partial \theta}{\partial z} \right)^2 = 0$$

may also be obtained by defining  $\theta$  and  $\alpha$  by means of the equations

$$[x - \xi(\theta, \alpha)]^2 + [y - \eta(\theta, \alpha)]^2 + [z - \zeta(\theta, \alpha)]^2 = 0,$$

$$\frac{\partial \xi}{\partial \alpha}(x - \xi) + \frac{\partial \eta}{\partial \alpha}(y - \eta) + \frac{\partial \zeta}{\partial \alpha}(z - \zeta) = 0.$$

I have not yet succeeded in finding a function  $u$  such that  $uf(\theta)$  is a solution of Laplace's equation.

BRYN MAWR COLLEGE,  
April, 1912.